Episode 9

Analyzing Collisions

ENGN0040: Dynamics and Vibrations

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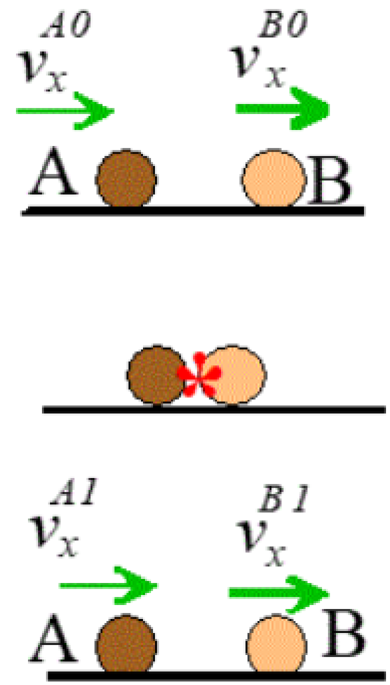
Topics for todays class

Analyzing Collisions

1. The Restitution Coefficient of a Collision
2. Straight Line Motion Collisions
3. 3D Collisions of frictionless bodies
4. Applications

4·5 Analyzing Collisions



Goal: Given V^{A0}, V^{B0}

Find V^{A1}, V^{B1}

Observations from experiment

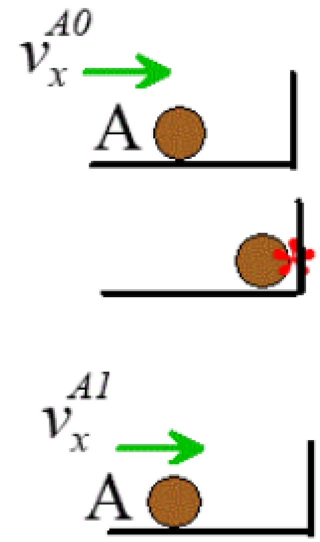
(1) Collision time is short (~½ ms)

(2) Large contact forces (~40 kN)

(3) Significant deformation occurs in colliding bodies

(4) Deformation may be reversible or irreversible

4.5.1 The restitution coefficient



For normal impact with a stationary surface, define

$$e = \frac{-V_x^{A1}}{V_x^{A0}}$$ (restitution coeft)

Experiments / simulations show

(1) e is approx constant for a given material (varies weakly with V_x^{A0})

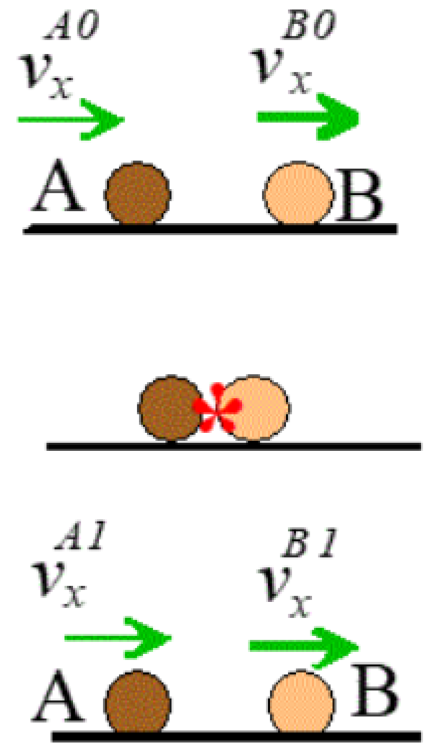
(2) $0 < e < 1$

$e = 1 \Rightarrow$ no energy lost "Elastic Collision"

$e = 0 \Rightarrow$ particle sticks "plastic" collision

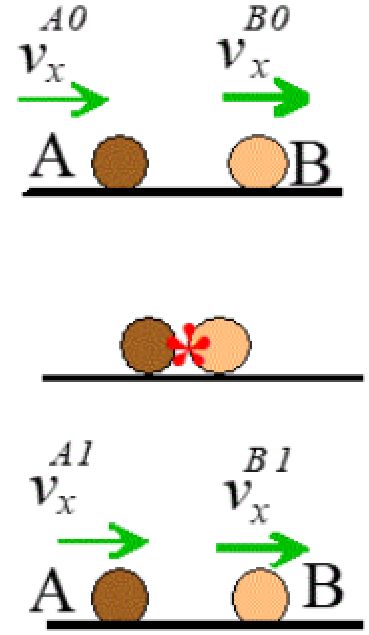
Straight line collision between moving objects

$$\boxed{e = -\frac{V_x^{A1} - V_x^{B1}}{V_x^{A0} - V_x^{B0}}}$$



e is nomally measured experimentally.

4.5.2 Straight line motion collision formulas



Momentum

System = two colliding objects
External impulse negligible since collision time is short

$$\Rightarrow \quad p_1^{TOT} = p_0^{TOT}$$

$$\Rightarrow \quad m_A V_x^{A1} + m_B V_x^{B1} = m_A V_x^{A0} + m_B V_x^{B0} \qquad (1)$$

Restitution

$$V_x^{A1} - V_x^{B1} = -e\left(V_x^{A0} - V_x^{B0}\right) \qquad (2)$$

$m_B(2)+(1) \Rightarrow$

$$(m_A+m_B)\,V_x^{A1} = (m_A - e m_B)\,V_x^{A0} + m_B(1+e)\,V_x^{B0}$$

$$\Rightarrow \boxed{V_x^{A1} = V_x^{A0} - \frac{m_B}{m_A+m_B}(1+e)\left(V_x^{A0}-V_x^{B0}\right)}$$

Similarly

$$\boxed{V_x^{B1} = V_x^{B0} + \frac{m_A}{m_A+m_B}(1+e)\left(V_x^{A0}-V_x^{B0}\right)}$$

Energy Loss

$$T_1 - T_0 = \frac{1}{2}m_A\left({V_x^{A1}}^2 - {V_x^{A0}}^2\right) + \frac{1}{2}m_B\left({V_x^{B1}}^2 - {V_x^{B0}}^2\right)$$

$$\Rightarrow \boxed{T_1 - T_0 = \frac{m_A m_B}{m_A+m_B}\,\frac{(e^2-1)}{2}\left(V_x^{A0}-V_x^{B0}\right)^2}$$

$e=1 \Rightarrow$ energy conserved

Example

$m_A = m_B \qquad V_x^{B0} = 0$

$$V_x^{A1} = V_x^{A0} - \frac{1}{2}(1+e)\,V_x^{A0} = \frac{(1-e)}{2}\,V_x^{A0}$$

$$V_x^{B1} = \frac{1}{2}(1+e)\,V_x^{A0}$$

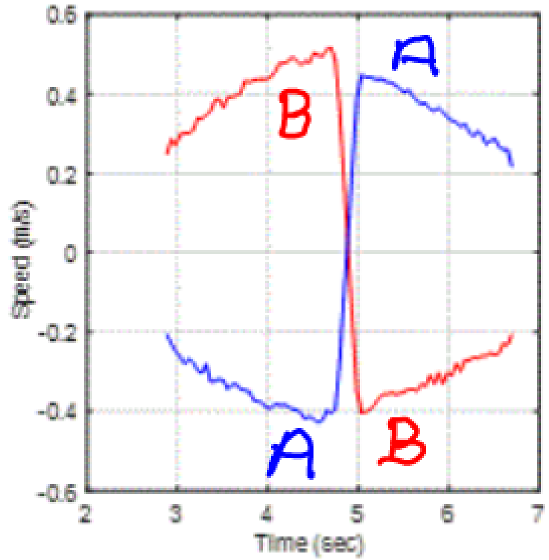
Hence:

For $e=1 \quad V_x^{A1}=0 \quad V_x^{B1} = V_x^{A0}$
(particles switch velocities)

For $e=0 \quad V_x^{A1} = V_x^{B1} = \dfrac{V_x^{A0}}{2}$

(particles stick)

4.5.3: Example: The graph plots the measured velocity of two 1m diameter granite spheres during a collision. Calculate the restitution coefficient.

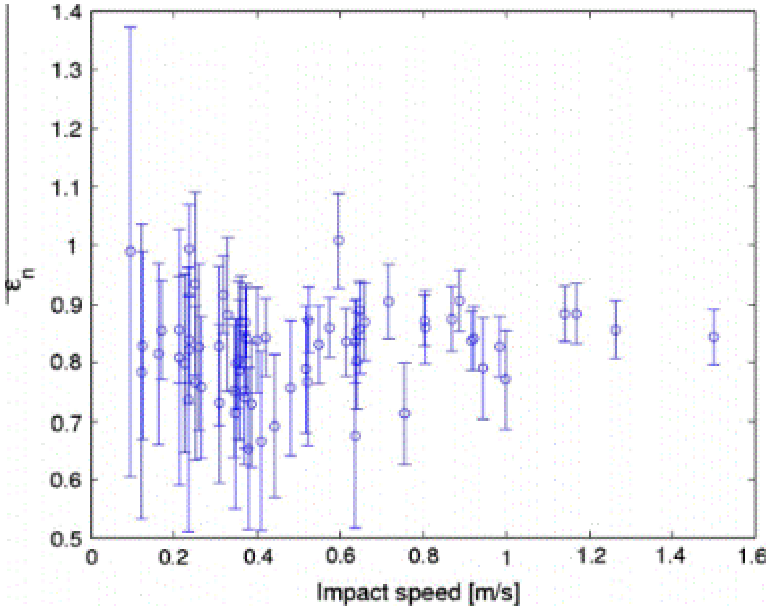


Use formula

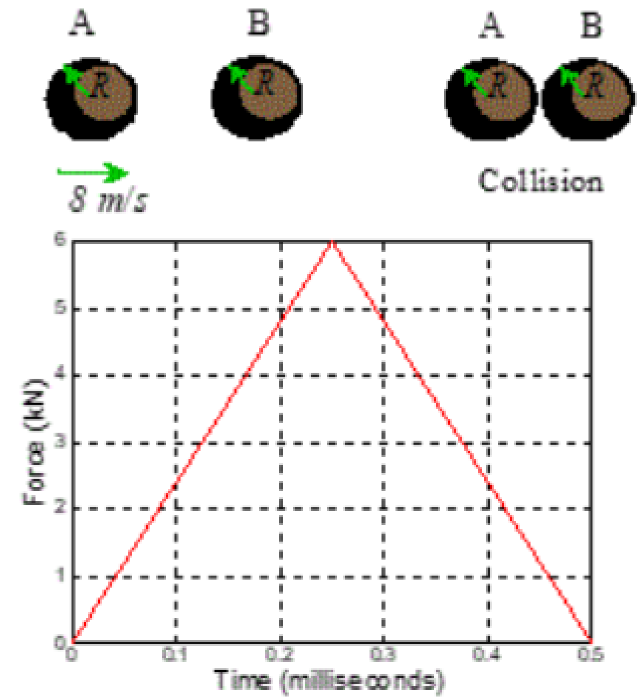
$$e = -\frac{(V_x^{A1} - V_x^{B1})}{(V_x^{A0} - V_x^{B0})}$$

$$e = -\frac{0.42 - (-0.4)}{(-0.4 - 0.5)} \Rightarrow \boxed{e = 0.91}$$

Data reported in publication



4.5.4: Example: A collision occurs between two spheres with mass 0.25kg. Before impact B is stationary and A moves with speed 8 m/s. The figure shows the magnitude of the force acting between them during the collision. Calculate the restitution coefficient.



Approach: (1) Impulse-momentum to find V_x^{A1} V_x^{B1}

(2) Restitution formula

(1) Impulse $\underset{\sim}{J} = \int_{t_0}^{t_1} F(t)\, dt$

$$= \frac{1}{2} \times 0.5\times10^{-3} \times 6\times10^{3} = 1.5Ns$$

Impulse - Momentum $\underset{\sim}{J} = \underset{\sim}{p}_1 - \underset{\sim}{p}_0$

for A: $-1.5\underset{\sim}{i} = 0.25\, V_x^{A1}\underset{\sim}{i} - 0.25 \times 8\underset{\sim}{i}$

$$\Rightarrow \quad V_x^{A1} = 2\underset{\sim}{i} \quad m/s$$

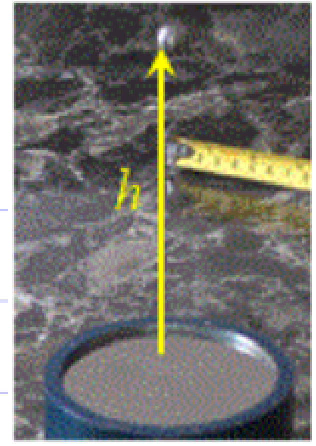
For B $\quad 1.5\underline{i} = 0.25\, V_x^{B'}\, \underline{i} - \underline{0}$

$$\Rightarrow \quad V_x^{B'} = 6\,\underline{i} \text{ m/s}$$

Restitution coeft

$$\boxed{e = -\frac{(2-6)}{(8-0)} = 0.5}$$

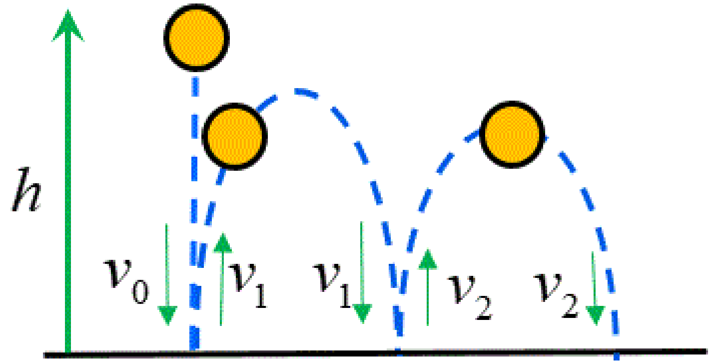
4.5.5: Example: A ball bearing is dropped from a height *h* onto a hard metal surface. It stops bouncing after a time interval *T*. Find a formula for the restitution coefficient.



① Speed just before first bounce

Energy Conservation $\Rightarrow \frac{1}{2} m V_0^2 = mgh$

$$\Rightarrow \quad V_0 = \sqrt{2gh}$$



② Speed after 1^{st} bounce $\quad V_1 = eV_0$

③ Speed after n^{th} bounce $\quad V_n = e^n V_0$

④ Time for n^{th} bounce

Impulse-momentum while airborne

$$\underline{J} = \underline{p}_1 - \underline{p}_0 \Rightarrow -mgT_n \underline{j} = -mV_n \underline{j} - mV_n \underline{j}$$

$$\Rightarrow \quad T_n = \frac{2V_n}{g} = \frac{2V_0}{g} e^n$$

⑤ Time for all bounces

$$T = \sum_{n=1}^{\infty} T_n = \frac{2V_0}{g} \underbrace{\sum_{n=1}^{\infty} e^n}_{= e/(1-e)}$$

$$\Rightarrow \frac{Tg}{2V_0} = \frac{e}{1-e}$$

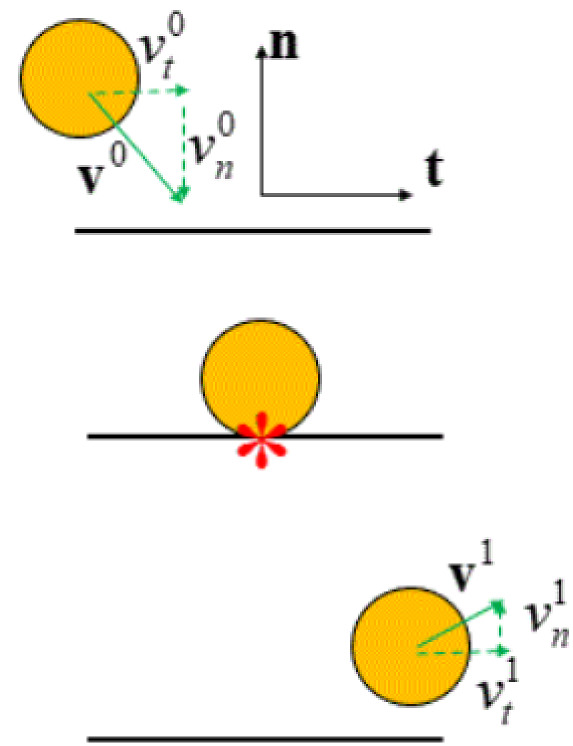
$$\Rightarrow \boxed{e = \left\{1 + \frac{2V_0}{Tg}\right\}^{-1} = \left\{1 + \frac{1}{T}\sqrt{\frac{8h}{g}}\right\}^{-1}}$$

For experiment $T = 28.06\ s$ $\quad h = 13\ cm$

$$\Rightarrow \boxed{e = 0.989}$$

4.5.6 3D restitution coefficient formula for frictionless collisions

Impact on a stationary surface



No friction $\Rightarrow$ no tangential impulse on particle

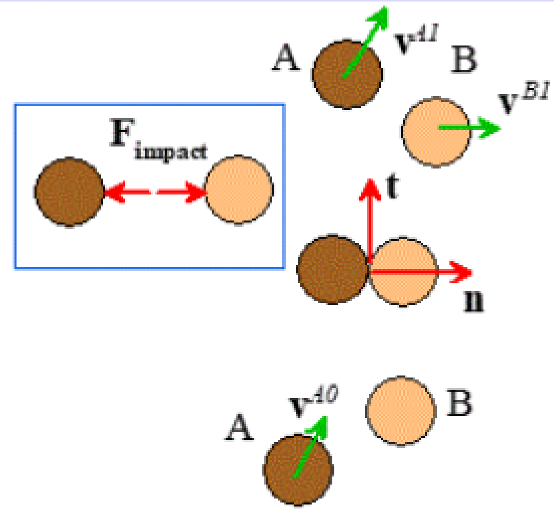
$$\Rightarrow \quad \boxed{V_t^1 = V_t^0}$$

Normal impact obeys 1-D formula

$$\boxed{V_n^1 = -e\,V_n^0}$$

Impact between moving particles

Formulas in $\{\underline{n}, \underline{t}\}$ components



Let $\underline{n}$ be parallel to line connecting centers of curvature at impact point

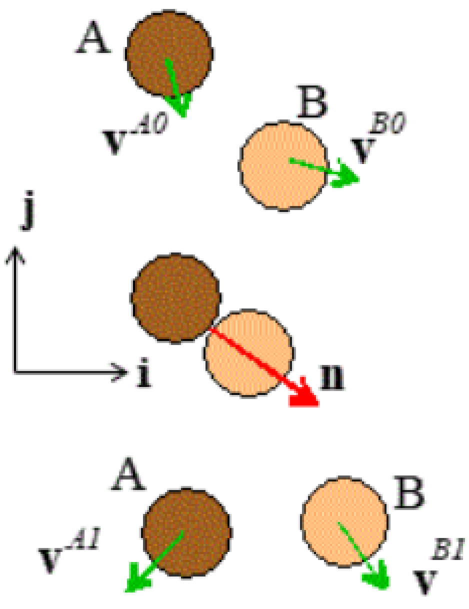
No friction $\Rightarrow$ no impulse parallel to $\underline{t}$

$$\Rightarrow \boxed{V_t^{A1} = V_t^{A0} \qquad V_t^{B1} = V_t^{B0} \quad \text{or} \quad V_t^{A1} - V_t^{B1} = V_t^{A0} - V_t^{B0}}$$

Normal components obey 1D formula

$$\boxed{V_n^{A1} - V_n^{B1} = -e\left(V_n^{A0} - V_n^{B0} \right)}$$

$\{\underline{i}, \underline{j}\}$ component formulas



$\underline{n}$: parallel to line connecting centers of curvature @ impact point

$$\underline{n} = n_x \underline{i} + n_y \underline{j}$$

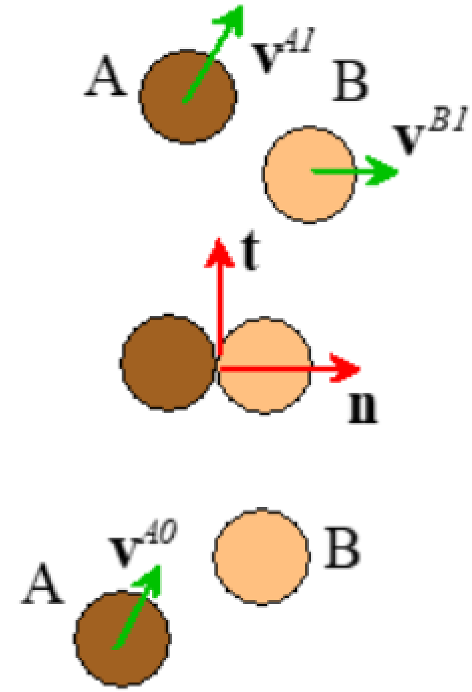
3D restitution formula

$$\boxed{\underline{V}^{A1} - \underline{V}^{B1} = \underline{V}^{A0} - \underline{V}^{B0} - \underbrace{(1+e)\left[\underline{V}^{A0} - \underline{V}^{B0}\right]\cdot\underline{n}}_{\text{magnitude}}\ \overbrace{\underline{n}}^{\text{Direction}}}$$

Formula embodies same physics as $\{\underline{n}, \underline{t}\}$ component formulas

Interpreting 3D restitution formula

Velocity components $\mathbf{v}^A = v_n^A \mathbf{n} + v_t^A \mathbf{t} \qquad \mathbf{v}^B = v_n^B \mathbf{n} + v_t^B \mathbf{t}$



Restitution Formula

$$\left(\mathbf{v}^{A1} - \mathbf{v}^{B1}\right) = \left(\mathbf{v}^{A0} - \mathbf{v}^{B0}\right) - (1+e)\left[\left(\mathbf{v}^{A0} - \mathbf{v}^{B0}\right)\cdot \mathbf{n}\right]\mathbf{n}$$

Dot with **n**

$$\left(\mathbf{v}^{A1} - \mathbf{v}^{B1}\right)\cdot \mathbf{n} = \left(\mathbf{v}^{A0} - \mathbf{v}^{B0}\right)\cdot \mathbf{n} - (1+e)\left[\left(\mathbf{v}^{A0} - \mathbf{v}^{B0}\right)\cdot \mathbf{n}\right]$$

$$\Rightarrow v_n^{A1} - v_n^{B1} = -e(v_n^{A0} - v_n^{B0})$$

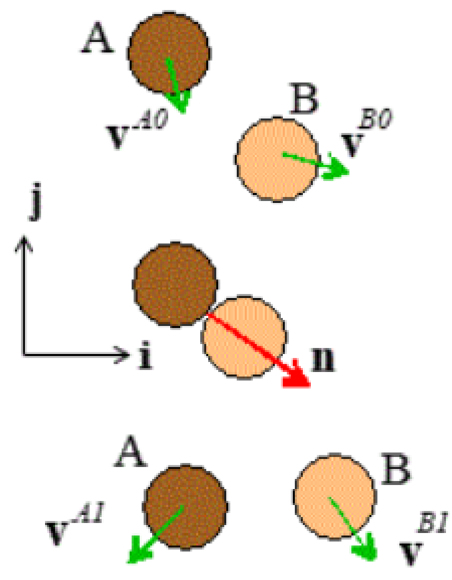
Normal velocity components related by 1-D formula

Dot with **t** $\quad \left(\mathbf{v}^{A1} - \mathbf{v}^{B1}\right)\cdot \mathbf{t} = \left(\mathbf{v}^{A0} - \mathbf{v}^{B0}\right)\cdot \mathbf{t}$

$$\Rightarrow v_t^{B1} - v_t^{A1} = v_t^{B0} - v_t^{A0}$$

No change in tangential relative velocities

4.5.7 3D collision formulas



Momentum:

$$m_A \underline{v}^{A1} + m_B \underline{v}^{B1} = m_A \underline{v}^{A0} + m_B \underline{v}^{B0}$$

Restitution

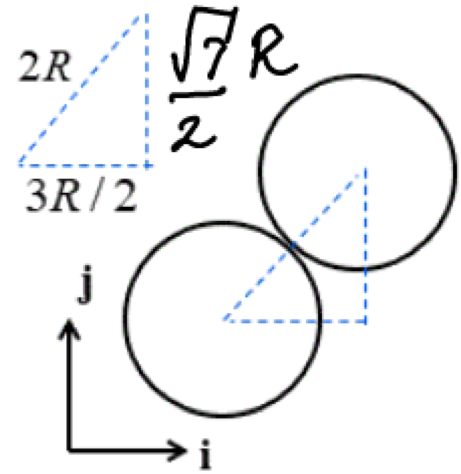
$$\underline{v}^{A1} - \underline{v}^{B1} = \underline{v}^{A0} - \underline{v}^{B0} - (1+e)\left[\underline{v}^{A0} - \underline{v}^{B0}\right]\cdot\underline{n}\ \underline{n}$$

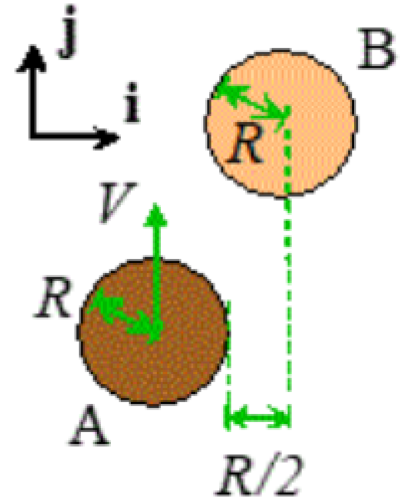
Solve for $\underline{v}^{A1}$ $\underline{v}^{B1}$:

$$\boxed{\underline{v}^{A1} = \underline{v}^{A0} - \frac{m_B}{m_A + m_B}(1+e)\left[\underline{v}^{A0} - \underline{v}^{B0}\right]\cdot\underline{n}\ \underline{n}}$$

$$\boxed{\underline{v}^{B1} = \underline{v}^{B0} + \frac{m_A}{m_A + m_B}(1+e)\left[\underline{v}^{A0} - \underline{v}^{B0}\right]\cdot\underline{n}\ \underline{n}}$$

4.5.8: Example: A and B have the same mass, and restitution coefficient e . Find their velocities after impact





(1) Find collision direction

$$\underline{n} = \left(\frac{3R}{2}\underline{i} + \frac{\sqrt{7}}{2}R\underline{j}\right) / 2R$$

$$= \left(\frac{3}{4}\underline{i} + \frac{\sqrt{7}}{4}\underline{j}\right)$$

Given: $\underline{v}^{A0} = V\underline{j}$ $\quad \underline{v}^{B0} = \underline{0}$

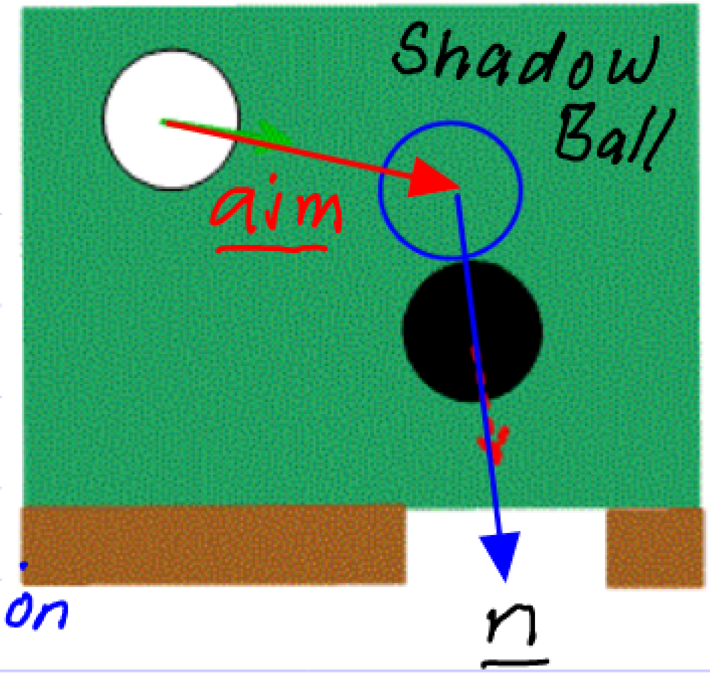
$$\Rightarrow \left[\underline{v}^{A0} - \underline{v}^{B0}\right] \cdot \underline{n} = V\frac{\sqrt{7}}{4}$$

Formulas:

$$\boxed{\underline{v}^{A1} = V\underline{j} - \frac{1}{2}(1+e)\frac{\sqrt{7}}{4}V\left(\frac{3}{4}\underline{i} + \frac{\sqrt{7}}{4}\underline{j}\right)}$$

$$\boxed{\underline{v}^{B1} = \frac{1}{2}(1+e)\frac{\sqrt{7}}{4}V\left(\frac{3}{4}\underline{i} + \frac{\sqrt{7}}{4}\underline{j}\right)}$$

4.5.9: Example: Where should you aim to pocket the black ball?

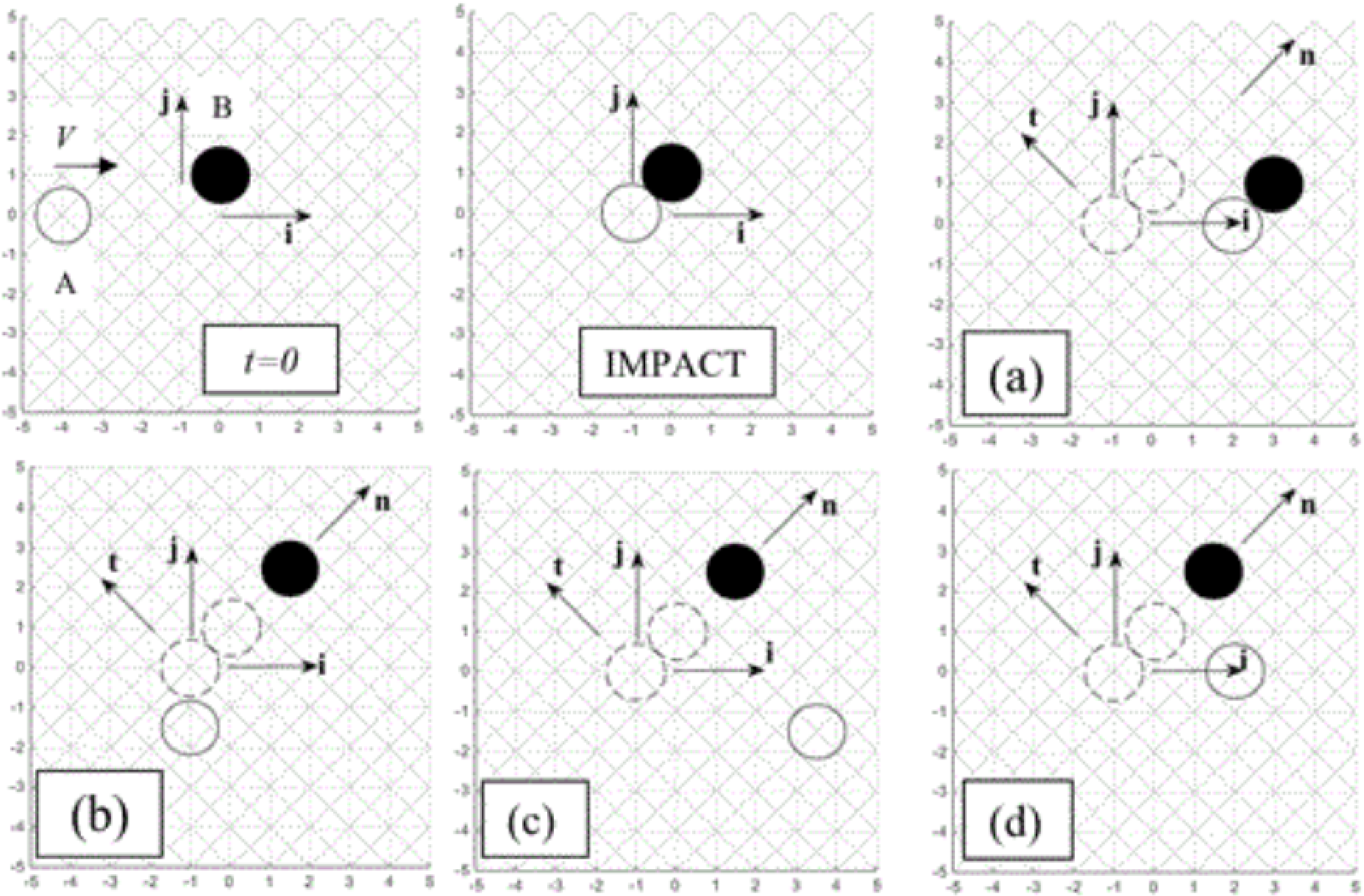


Formula:

$$\underline{v}^{B1} = \cancelto{=0}{\underline{v}^{B0}} + \underbrace{\frac{1}{2}(1+e)[\underline{v}^{A0} - \underline{v}^{B0}]\cdot\underline{n}}_{\text{magnitude}}\ \underbrace{\underline{n}}_{\text{Direction}}$$

Black ball will move along collision direction $\underline{n}$

"Shadow Ball" method: visualize a "shadow ball" behind object ball in line with pocket and aim at its center



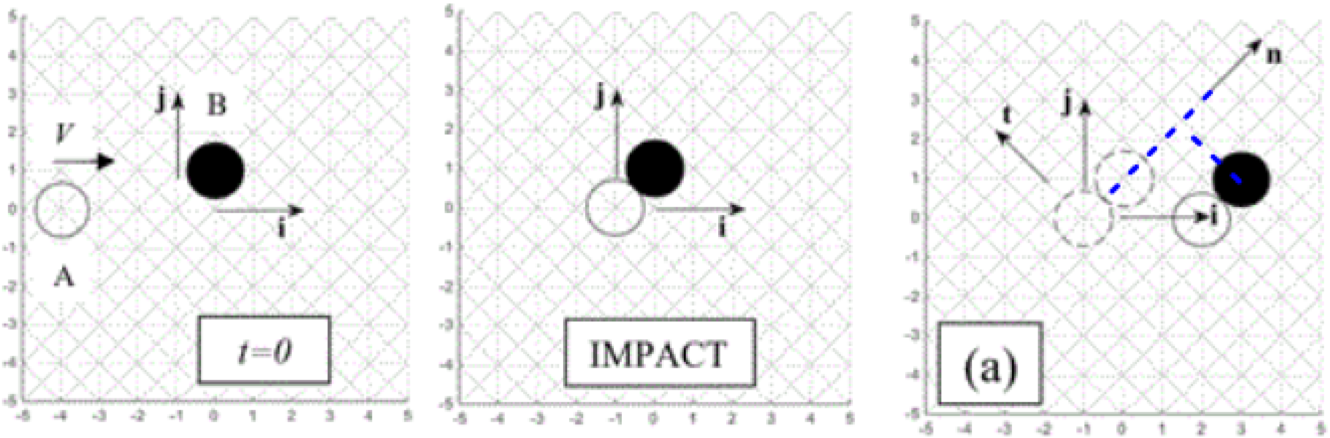
4.5.10: Example: Two spheres with equal mass and restitution coefficient *e=0* collide. For each of (a)-(d) identify whether

Total Momentum is conserved in the **j** direction

Momentum of B is conserved in the **t** direction

The restitution formula is satisfied in the **n** direction

Hence, pick the correct positions of the spheres after impact



Total Momentum is conserved in the **j** direction T F

Momentum of B is conserved in the **t** direction T F

The restitution formula is satisfied in the **n** direction T F

Initial momentum $p_0^{TOT} = m\bar{V}\underline{i}$

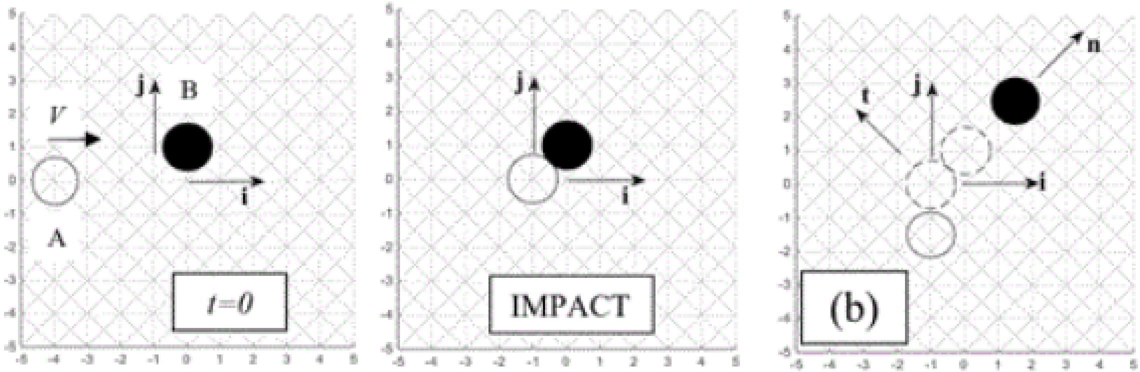
A, B move only in $\underline{i}$ direction after impact

Initial momentum of B $p_0^B = \underline{0}$

After impact B has moved in both $\{\underline{n}\}$ and $\{\underline{t}\}$ direction

Restitution formula $V_n^{A'} - V_n^{B'} = 0$

A & B move same distance in $\underline{n}$ direction after impact

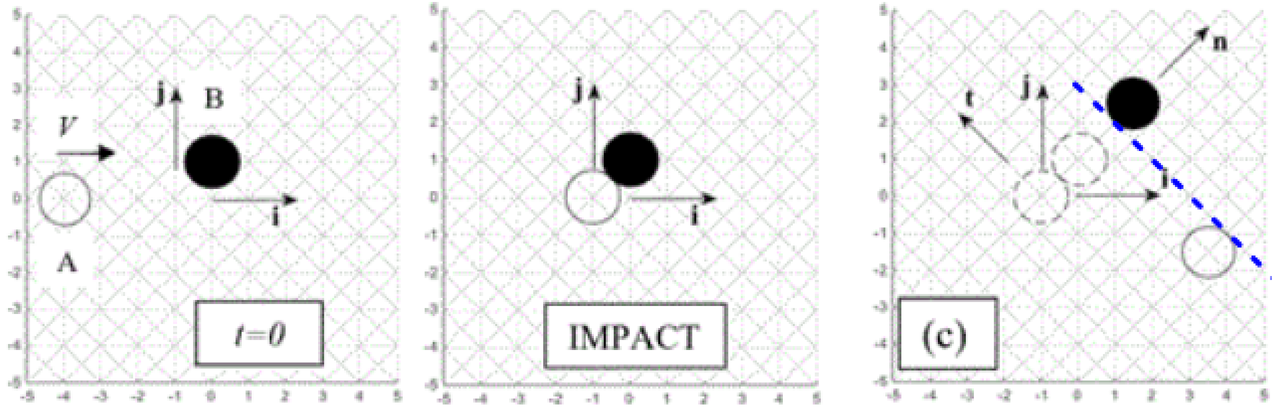


Total Momentum is conserved in the **j** direction	T	F
Momentum of B is conserved in the **t** direction	T	F
The restitution formula is satisfied in the **n** direction	T	F

Final momentum: note A&B move equal & opposite distances after collision in j dir.
$\Rightarrow$ j component of $p^{TOT} = 0$

B moves parallel to $\underline{n}$ $\Rightarrow$ $\underline{t}$ component of $p^{B'} = 0$

A & B have different speeds in $\underline{n}$ direction after collision $\Rightarrow$ $V_n^{A'} \neq V_n^{B'}$



Total Momentum is conserved in the **j** direction T F

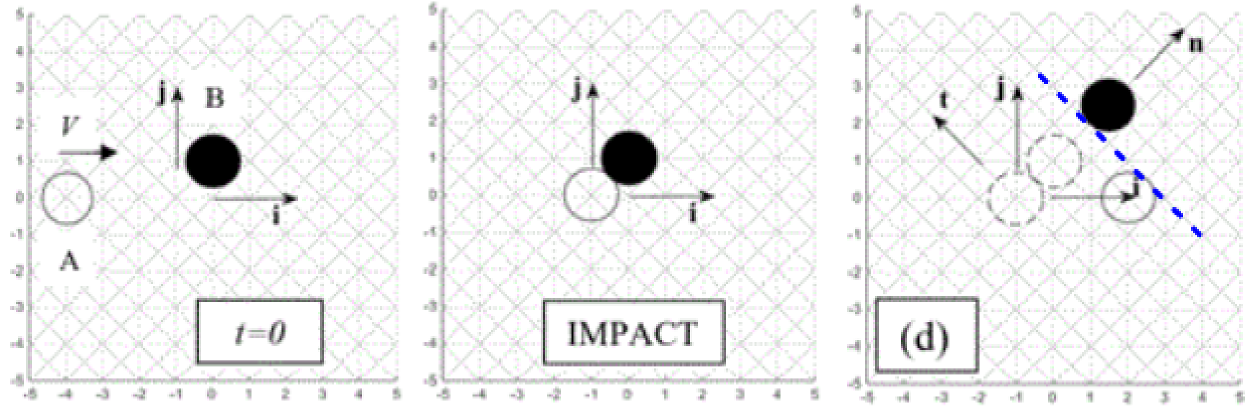
Momentum of B is conserved in the **t** direction T F

The restitution formula is satisfied in the **n** direction T F

A&B have equal & opposite displacements in j dir after collision

B moves parallel to $\underline{n}$ after collision

A&B have equal displacements in $\underline{n}$ direction after collision $\Rightarrow V_n^{A'} = V_n^{B'}$



Total Momentum is conserved in the **j** direction T F

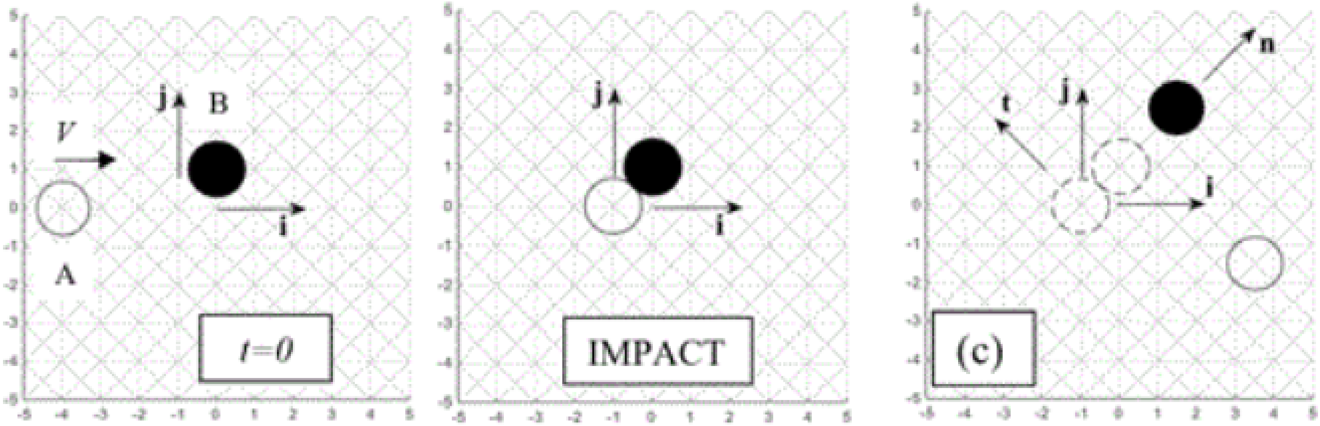
Momentum of B is conserved in the **t** direction T F

The restitution formula is satisfied in the **n** direction T F

A moves parallel to $\underline{i}$, B has j component of displacement after collision $\Rightarrow$ $p_y^{TOT} \neq 0$

B moves parallel to $\underline{n}$ after collision

A & B have equal displacements after collision



Total Momentum is conserved in the **j** direction T F
Momentum of B is conserved in the **t** direction T F
The restitution formula is satisfied in the **n** direction T F

The correct figure !